Network Geometry

Marián Boguñá

Department of Condensed Matter Physics, University of Barcelona
University of Barcelona Institute of Complex Systems

M. Ángeles Serrano
Universitat de Barcelona

Dmitri Krioukov
Northeastern University
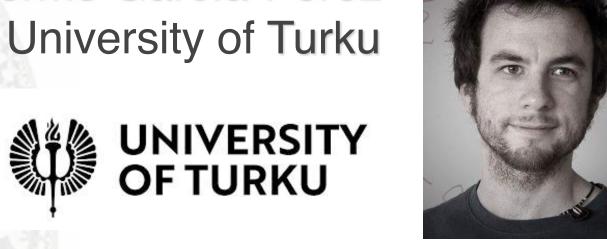
Fragkiskos Papadopoulos
Cyprus University of Technology

Guillermo García-Pérez
University of Turku

Antoine Allard
UNIVERSITÉ LAVAL
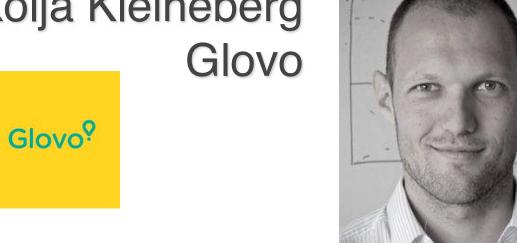
Maksim Kitsak
Delft University of Technology
TUDelft

Konstantin Zuev
Caltech

Pedro Almagro
Universidad de Sevilla

Kaj-Kolja Kleineberg
Glovo

Muhua Zheng
Jiangsu University

Amin Vahdat
UCSD
UC San Diego
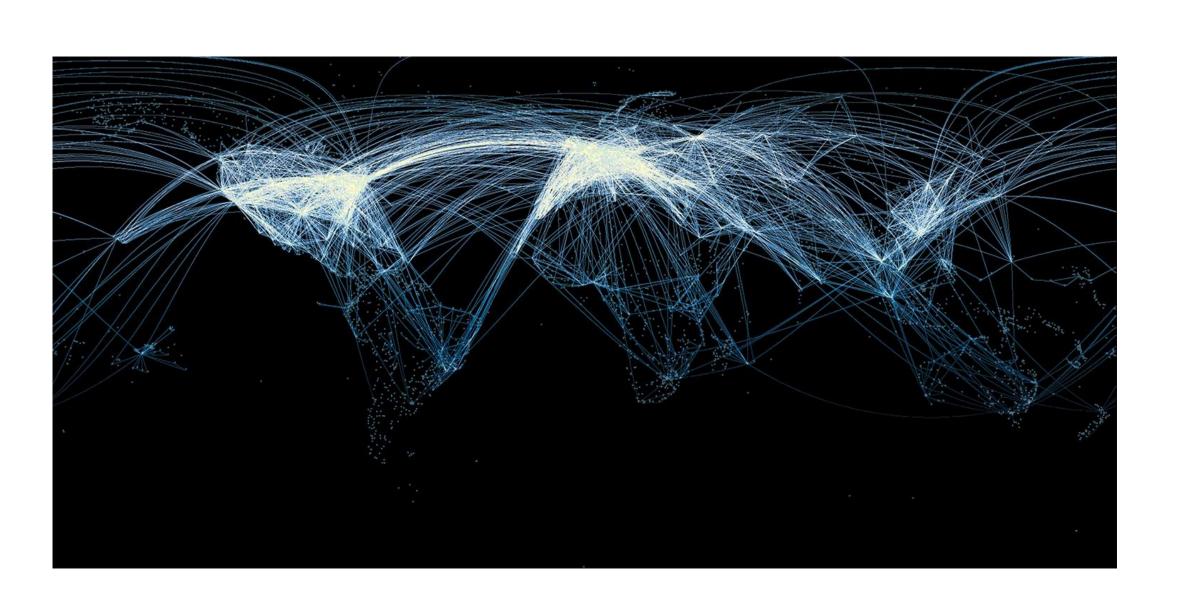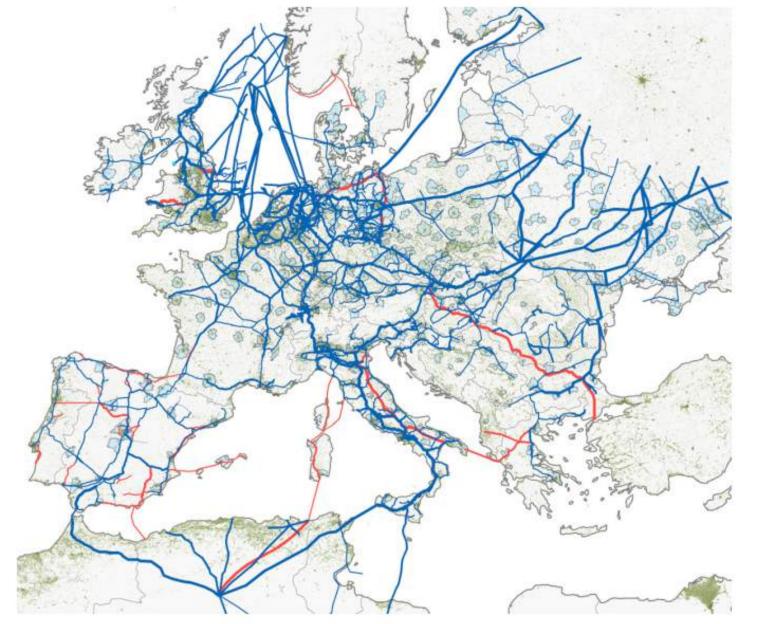
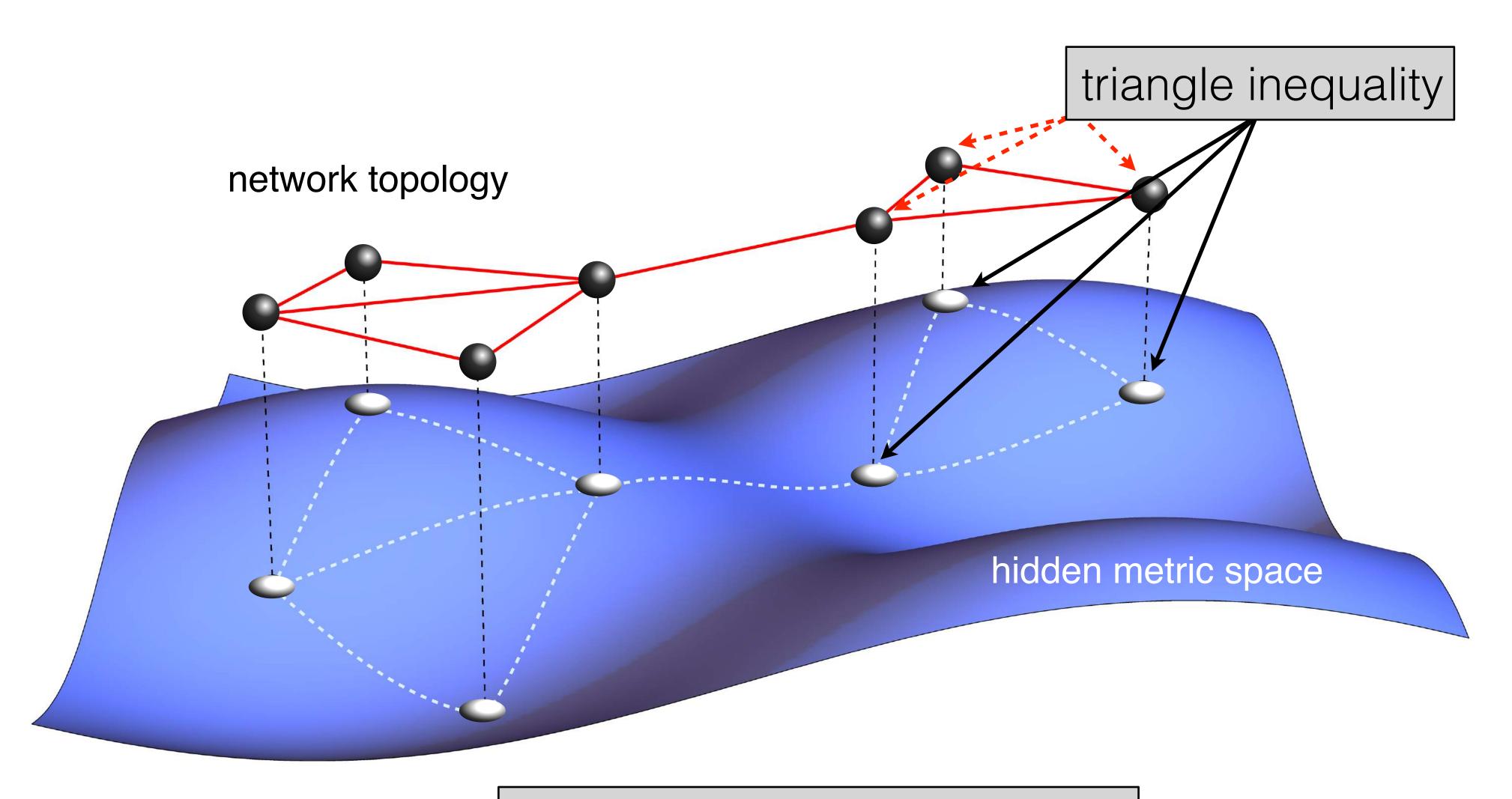Networks can be described as geometric objects: Links' existence depends on distances between nodes

in some networks geometry is explicit, power grids, airport networks, road networks, etc
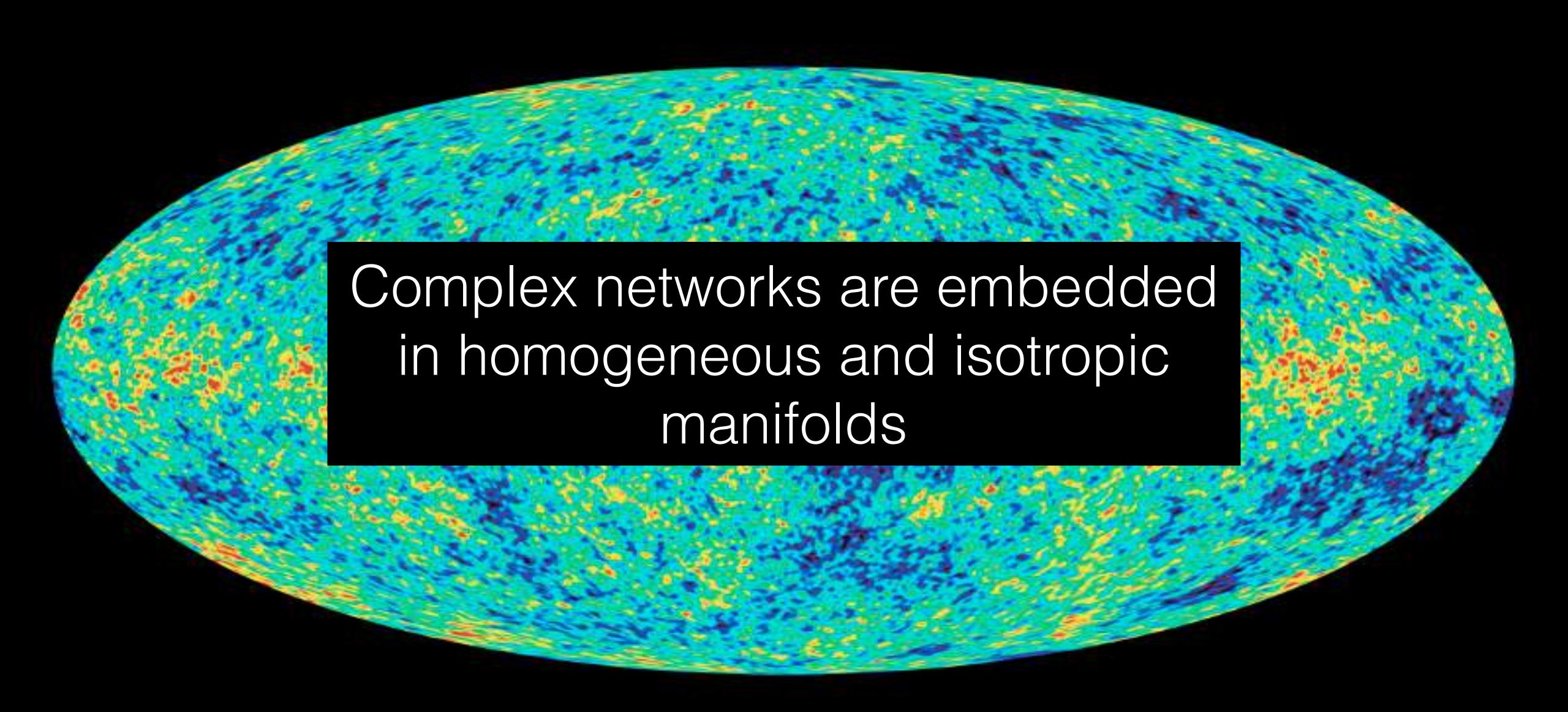
in others it is implicit: similarity space. It can be detected analyzing the clustering coefficient

triangle inequality

network topology

hidden metric space
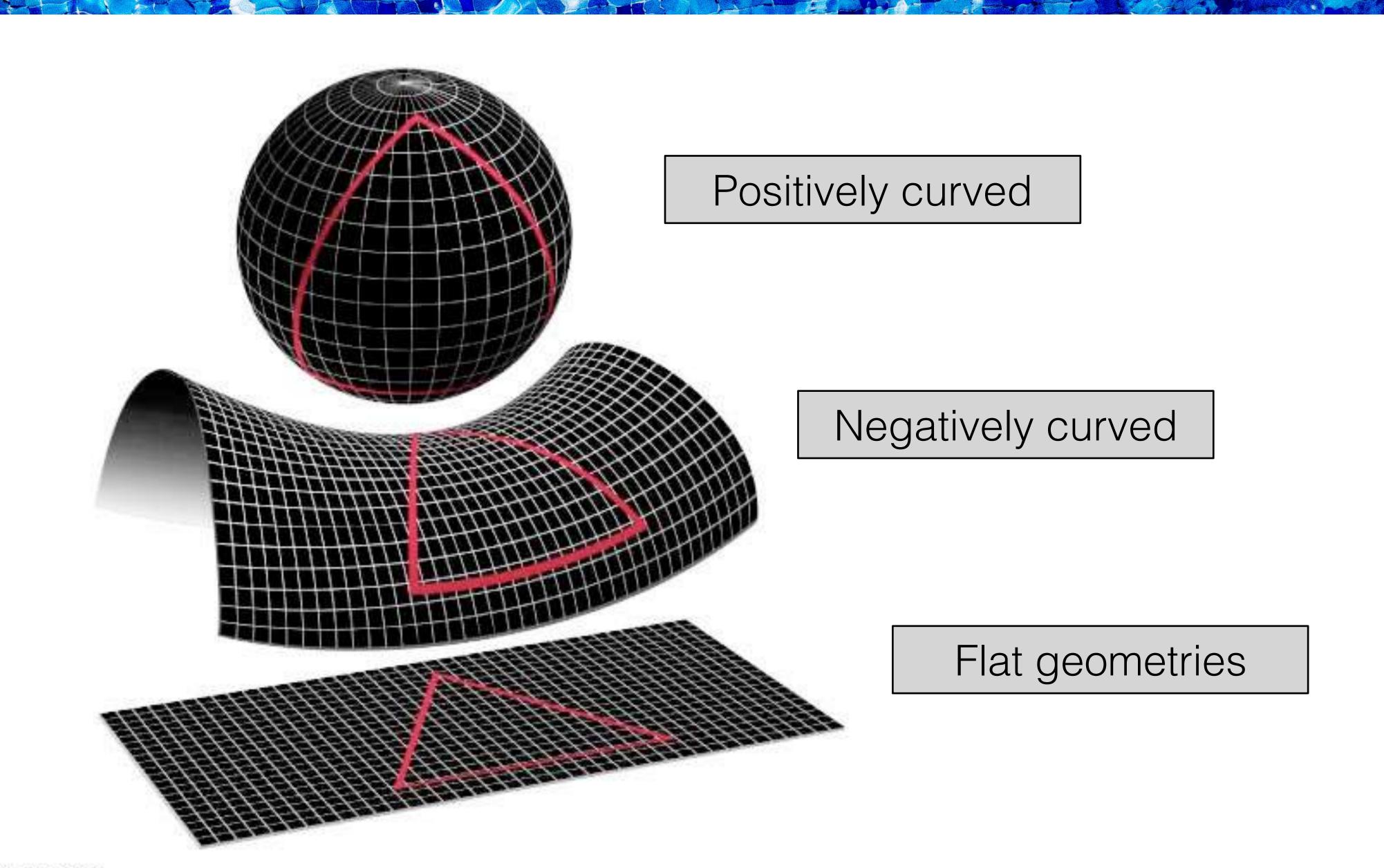
Clustering is just a byproduct
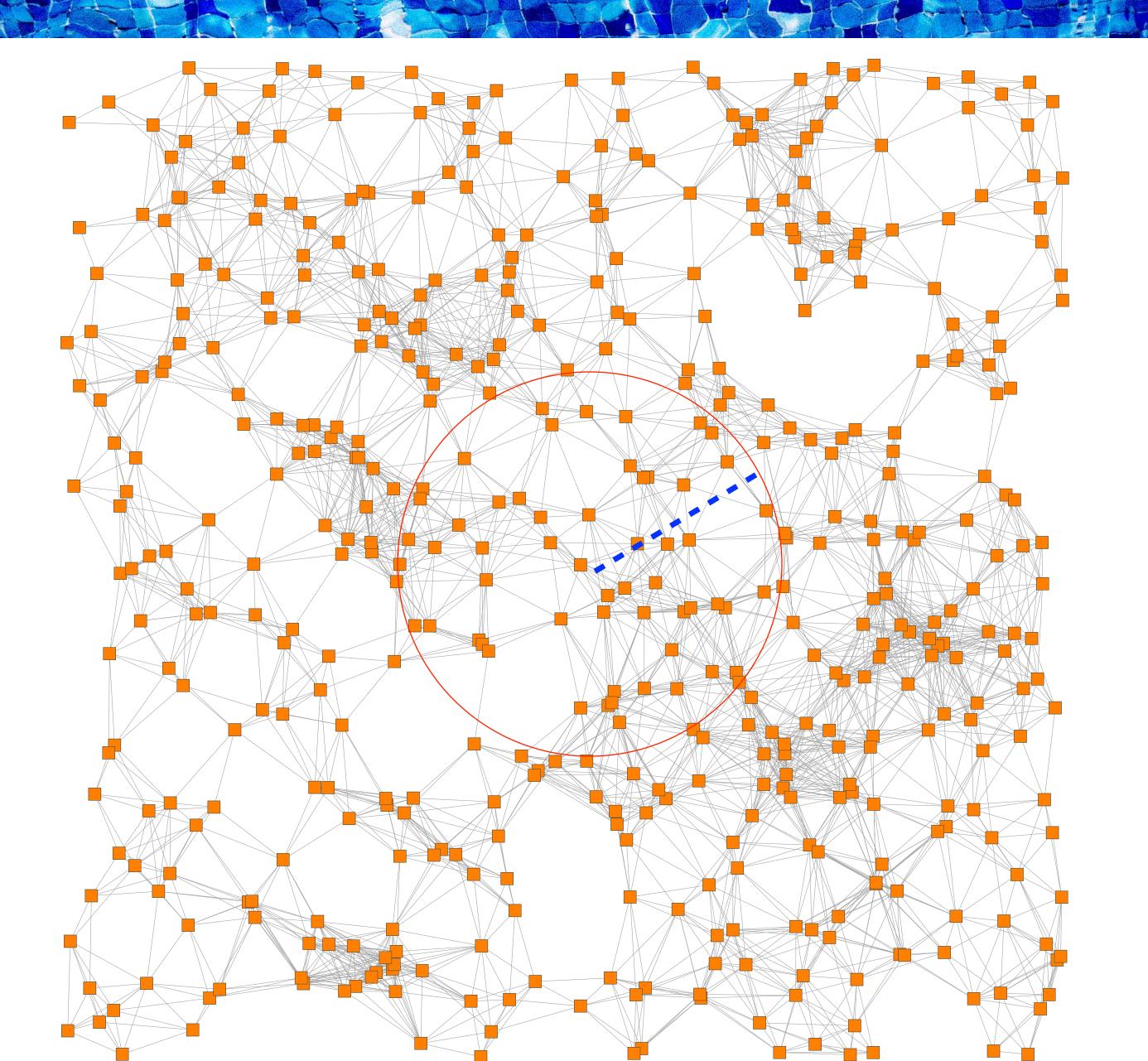
Complex networks are embedded in homogeneous and isotropic manifolds

Positively curved

Negatively curved

Flat geometries

Distribute points in a plane using a Poisson process or whatever you like

Connect points that are below a critical distance

problems

The generated graphs are not small-worlds

Graphs are homogeneous

Not a good model for real systems!!

Small-world effect

It can be rephrased as

$$l \sim \ln N$$

$$N(r) \sim e^r$$

Average shortest path length

Number of nodes

Number of nodes within a ball of radius r
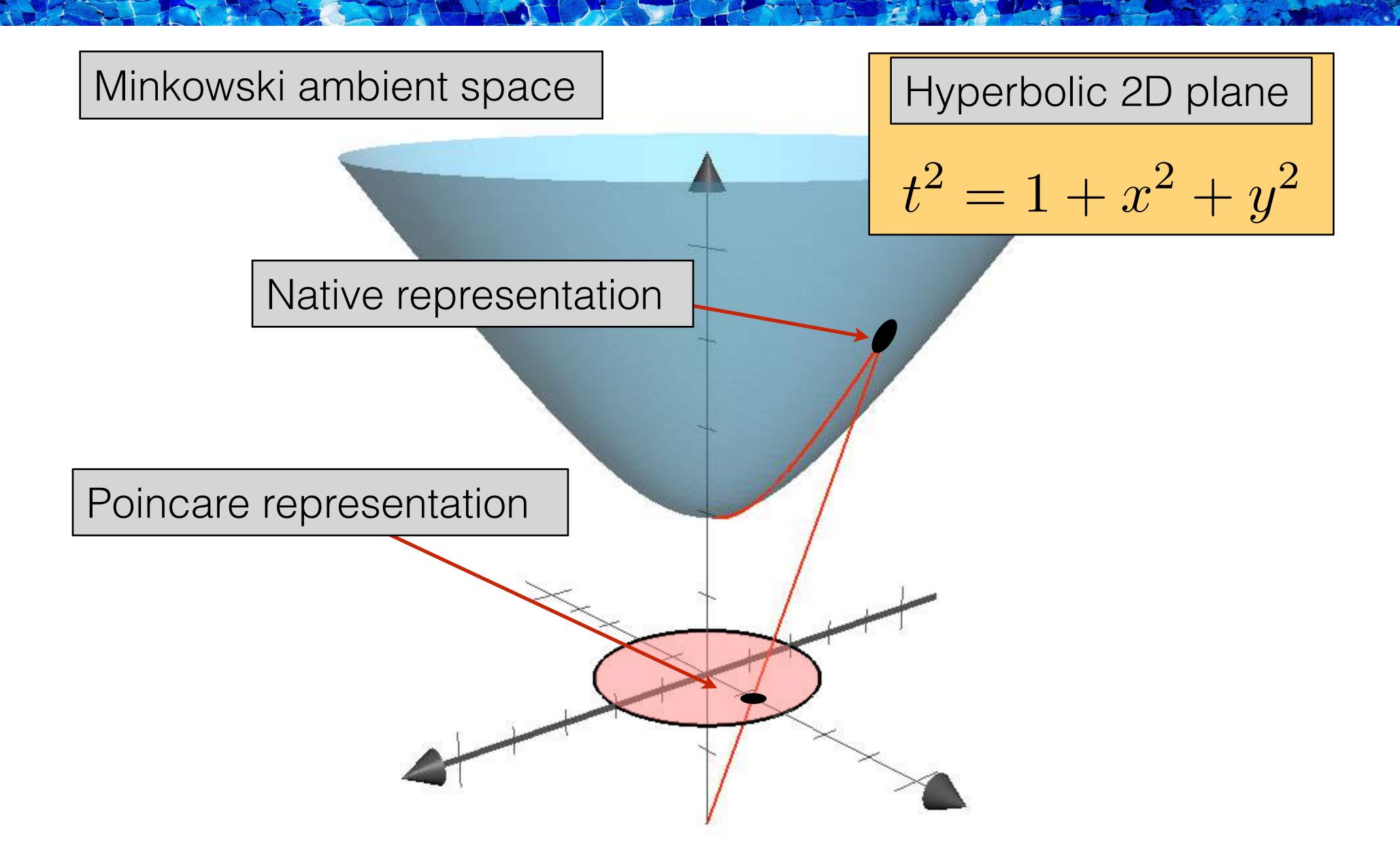
Euclidean

$$N(r) \sim r^2$$

Spherical

$$N(r) \sim 1 - \cos r$$

Hyperbolic

$$N(r) \sim \cosh r - 1$$
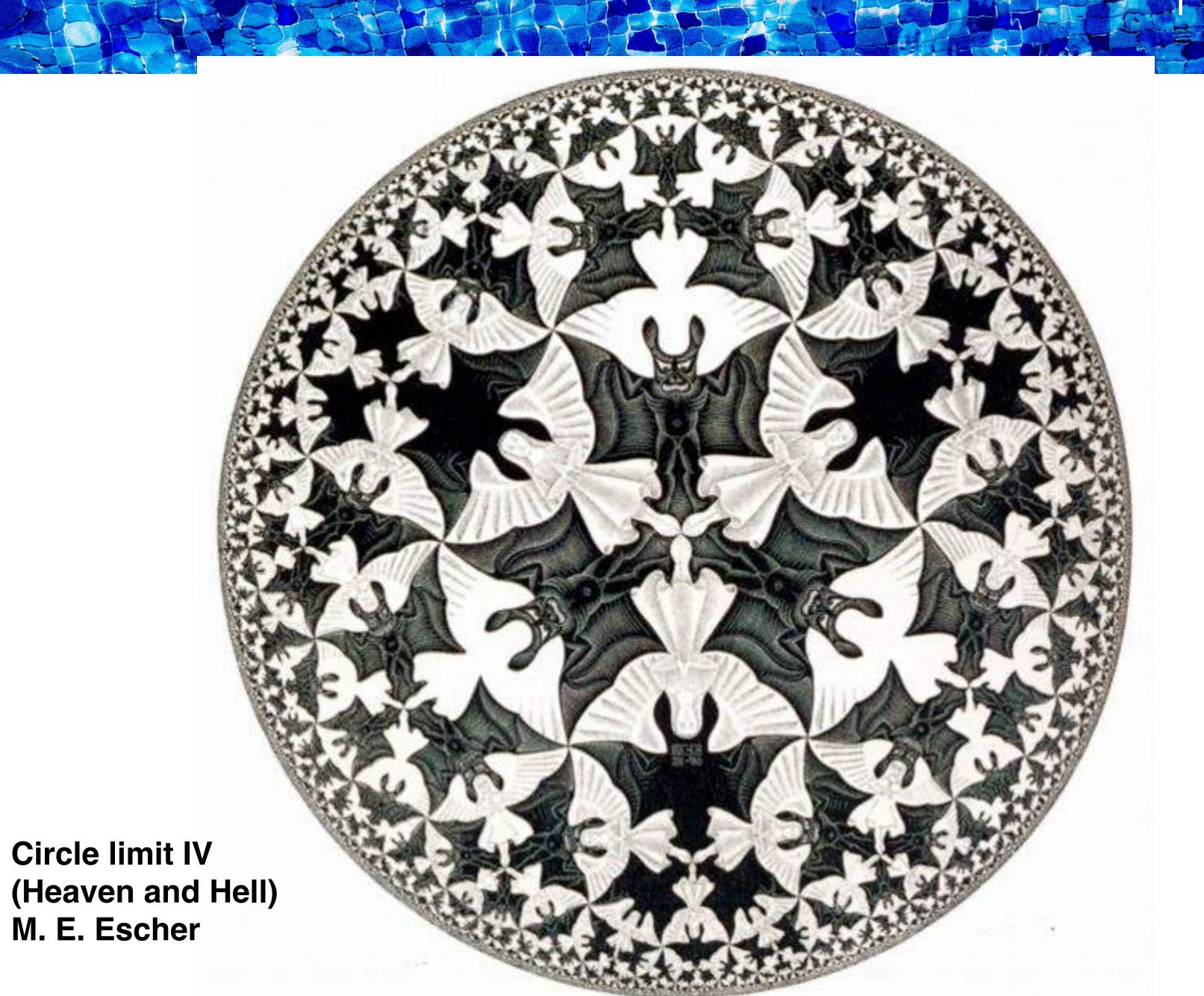
Minkowski ambient space

Hyperbolic 2D plane

$$t^2 = 1 + x^2 + y^2$$

Native representation

Poincare representation

**Circle limit IV
(Heaven and Hell)
M. E. Escher**

ric random graphs in $\mathbf{H}^2$

Scale-free, small-world, and highly clustered networks!!!

$$\rho(r) = \frac{\sinh r}{\cosh R - 1} \approx e^{r-R} \sim e^r$$

connection probability

$$p(x) = \Theta(R - x)$$

Gravity law in Euclidean or spherical geometry

$$\text{Prob}(m, m', d) = r\left(\frac{d}{mm'}\right)$$

$+$

power law distribution of masses

$$\rho(m) = (\gamma - 1)\frac{m_0^{\gamma-1}}{m^\gamma}$$

$=$

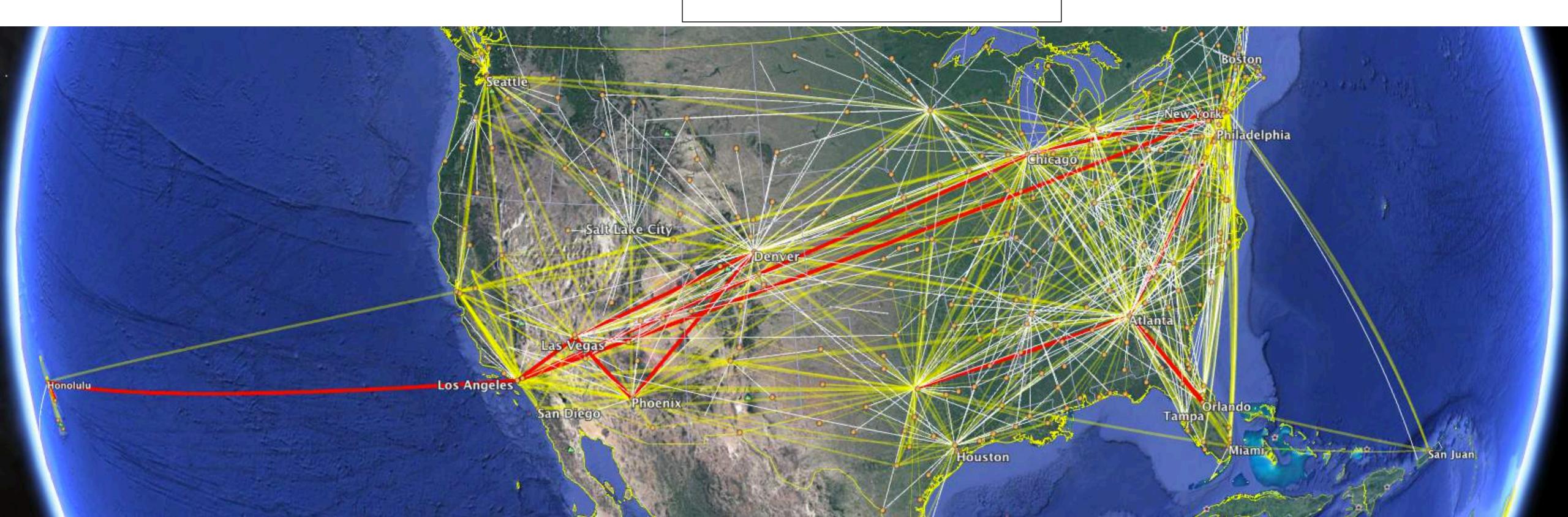**effective hyperbolic geometry**

Gravity law in Euclidean or spherical geometry

$+$

power law distribution of masses

$=$

effective hyperbolic geometry

$$\text{Prob}(m, m', d) = r\left(\frac{d}{mm'}\right)$$

$$\rho(m) = (\gamma - 1)\frac{m_0^{\gamma-1}}{m^\gamma}$$

$$r = R - \frac{2}{\zeta}\ln\left[\frac{m}{m_0}\right]$$

$$r\left(\frac{d}{mm'}\right) = r\left(\frac{\Delta\theta N}{2\pi mm'}\right) = \hat{r}\left(e^{\frac{\zeta}{2}\left(r+r'+\frac{2}{\zeta}\ln\frac{\Delta\theta}{2}-R\right)}\right)$$

S¹ model

$$\hat{r}\left(e^{\frac{\zeta}{2}(\underbrace{r+r'+\frac{2}{\zeta}\ln\frac{\Delta\theta}{2}-R})}\right)$$

$$x = \boxed{r + r' + \frac{2}{\zeta}\ln\frac{\Delta\theta}{2}}$$

$$\boxed{\hat{r}\left(e^{\frac{\zeta}{2}(x-R)}\right)}$$

$$\hat{r}(z)\begin{cases} \text{cte} & z \ll 1 \\ 0 & z \gg 1 \end{cases}$$

H² model

the hyperbolic distance is very well approximated by

$$\boxed{x = r + r' + \frac{2}{\zeta}\ln\sin\frac{\Delta\theta}{2}}$$
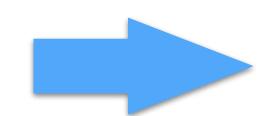
$$\boxed{p(x) = \Theta(R - x)}$$

Holographic principle

S¹ model and H² model are isomorphic

Gravity law in Euclidean or spherical geometry

$$\text{Prob}(m, m', d) = r\left(\frac{d}{mm'}\right)$$

For any integrable function *r(x)*

➡️ $$\bar{k}(m) \propto m$$

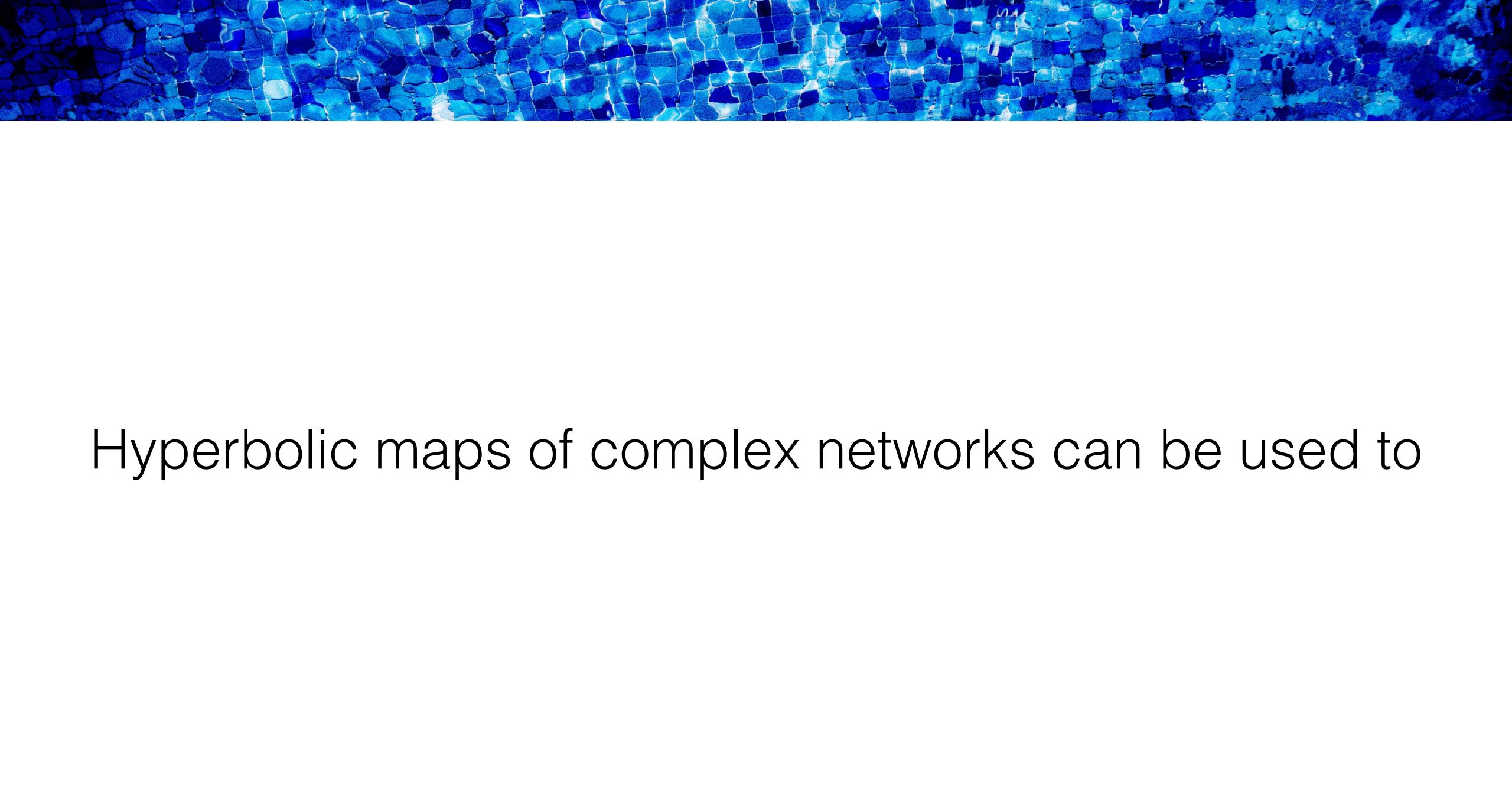So this is the expected degree of the node. From now on, let's call it κ

Maximally random ensembles of geometric graphs with finite clustering, small-worlds, and uncorrelated at the hidden level
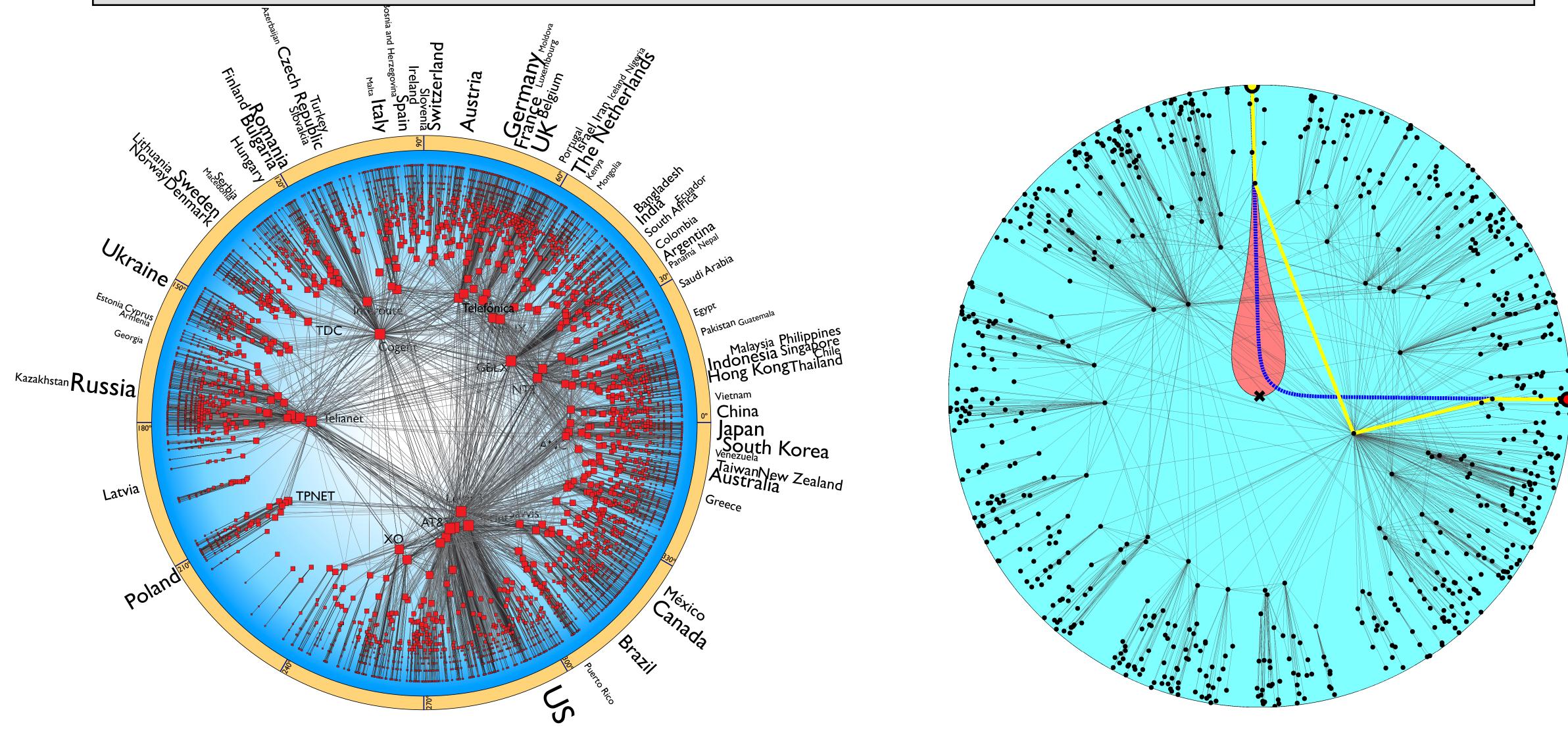
This exponent controls the level of clustering

$$\text{Prob}(\kappa_i, \kappa_j, d_{ij}) = r(\chi_{ij}) = \frac{1}{1 + \chi_{ij}^{\beta}} \quad \text{with} \quad \chi_{ij} = \frac{d_{ij}}{\mu \kappa_i \kappa_j}$$
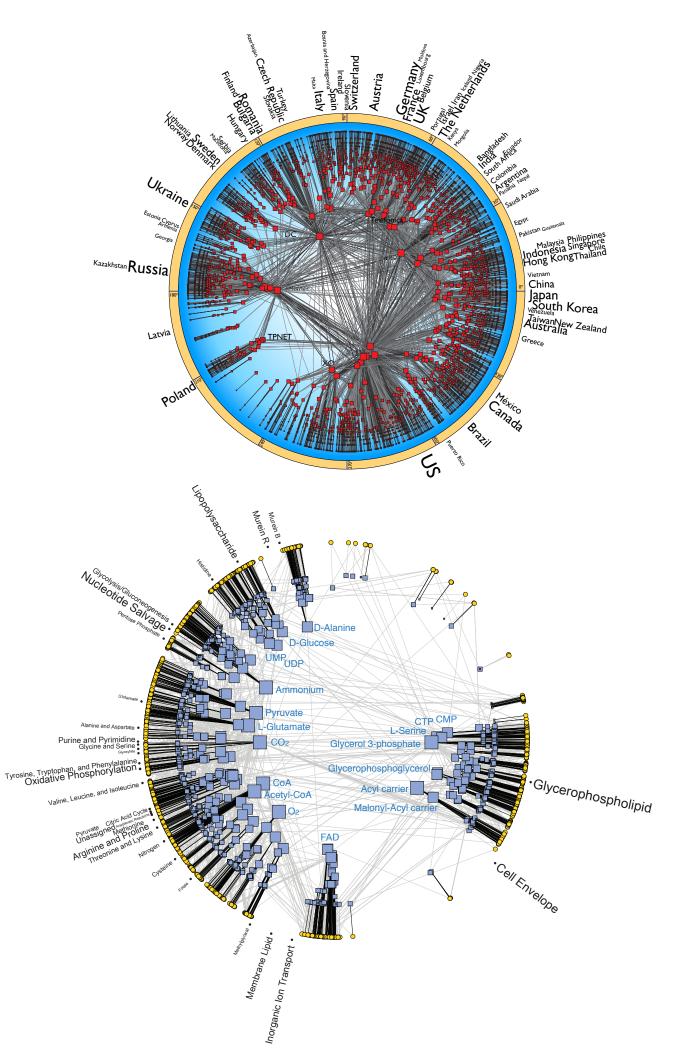
Real networks in the hyperbolic plane

The Internet

Metabolic networks

The world trade web

Airport network

Human connectome

Africa   Europe   Oceania
Asia   N. America   S. America

Hyperbolic maps of complex networks can be used to

Sustaining the Internet with hyperbolic mapping
M. Boguñá, F. Papadopoulos, and D. Krioukov
Nature Communications 1, 62 (2011)

Navigating the Internet

Navigate the Internet without a global knowledge of the network topology

Uncovering the hidden geometry behind metabolic networks
M. A. Serrano, M. Boguñá, and F. Sagués
Molecular BioSystems 8, 843-850 (2012)
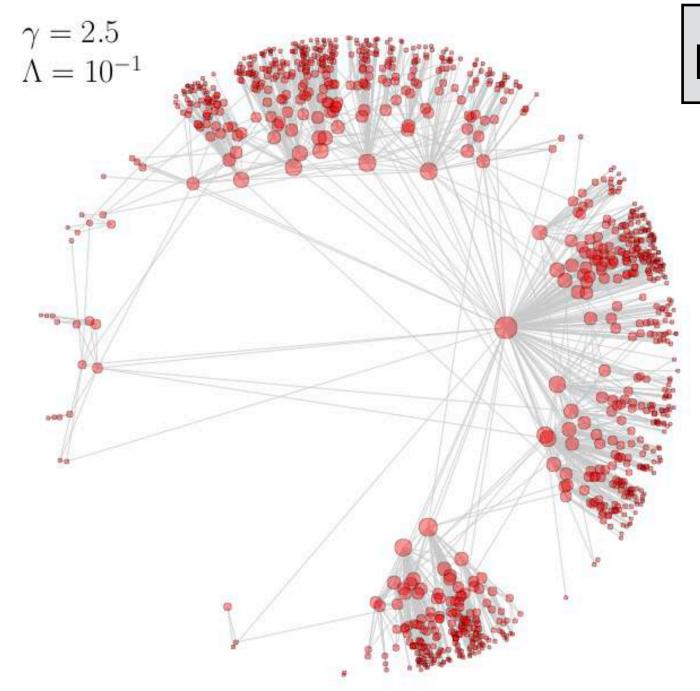
Find communities related to the nodes distribution in the similarity space
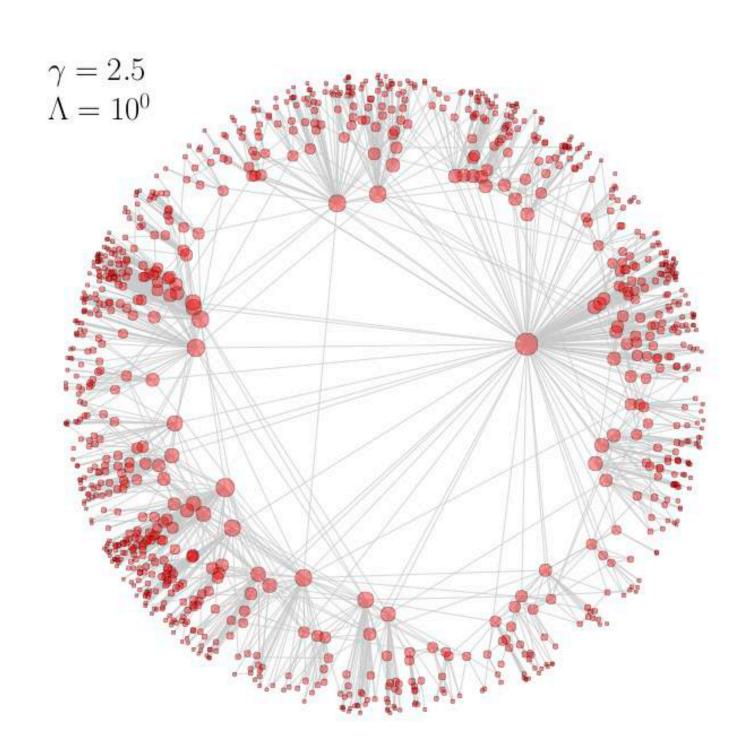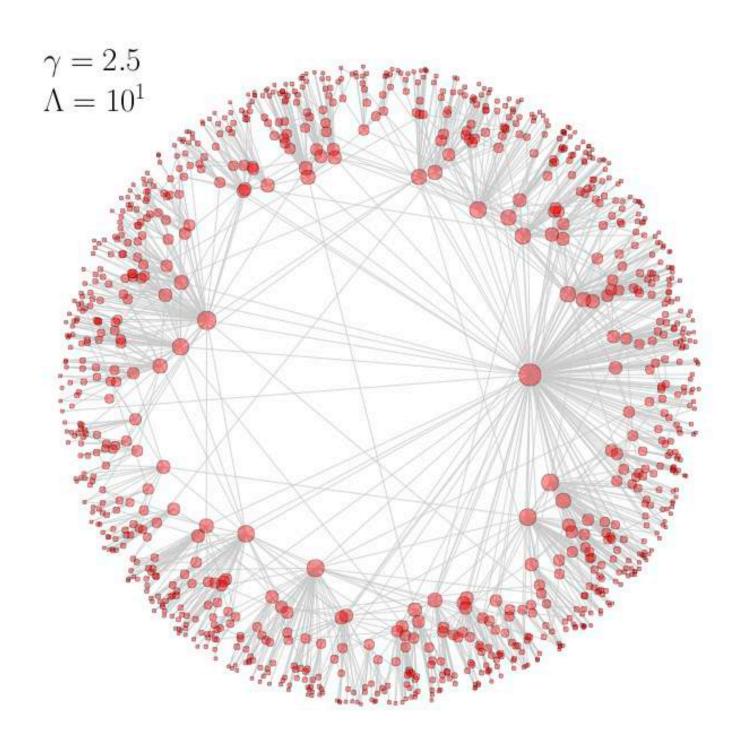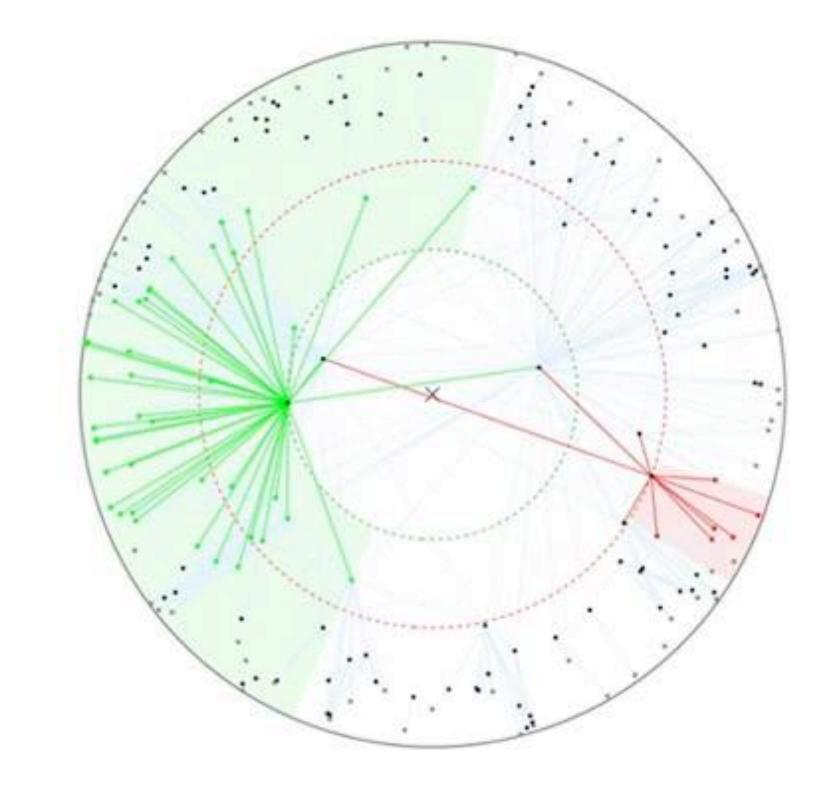
models generating communities

$\gamma = 2.5$
$\Lambda = 10^{-1}$

$\gamma = 2.5$
$\Lambda = 10^{0}$

$\gamma = 2.5$
$\Lambda = 10^{1}$
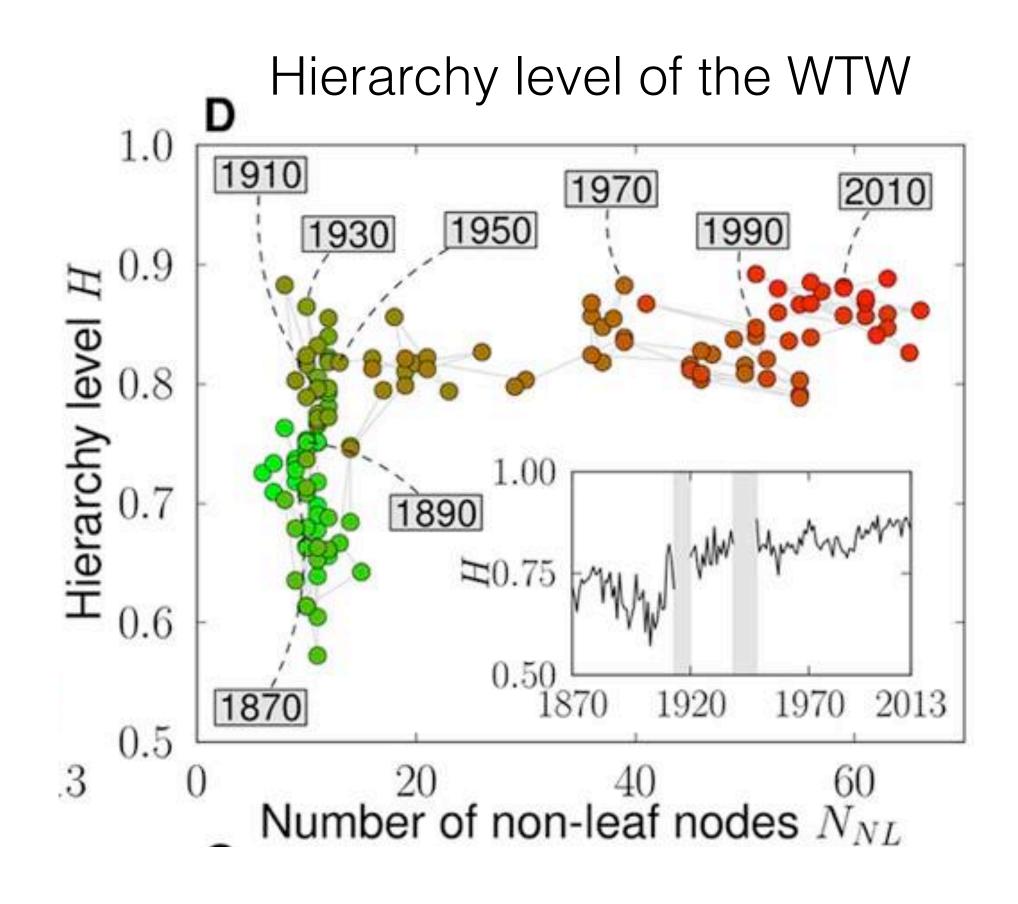
Define hierarchies in complex networks

B Hierarchy level

Hierarchy level of the WTW
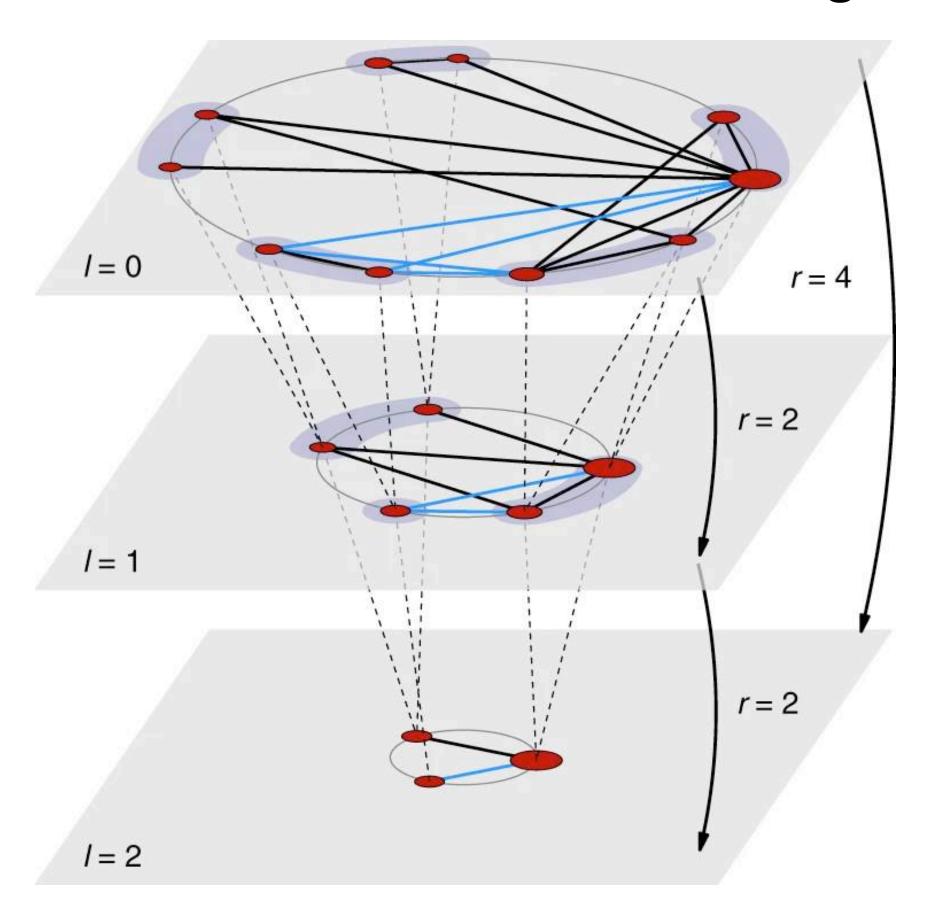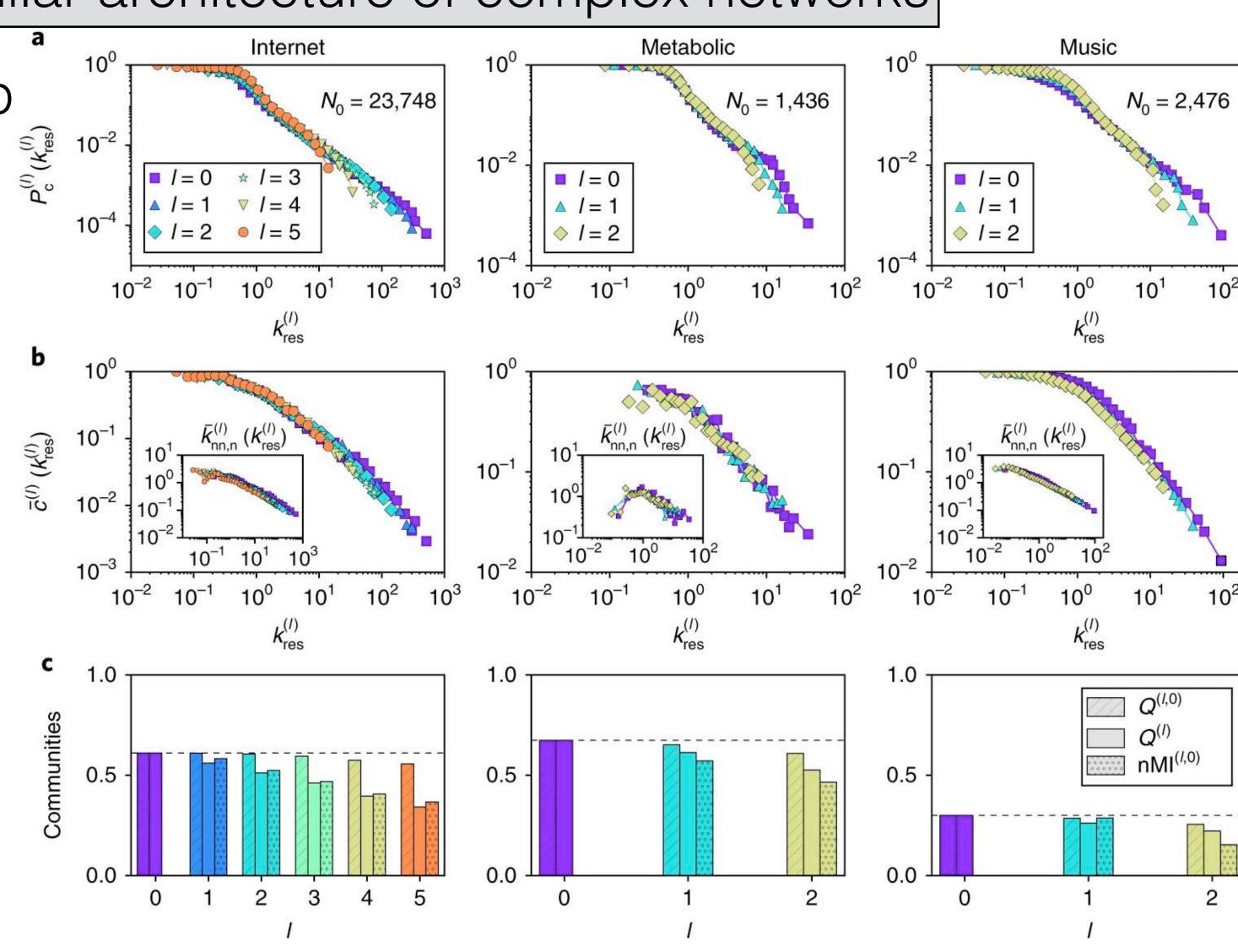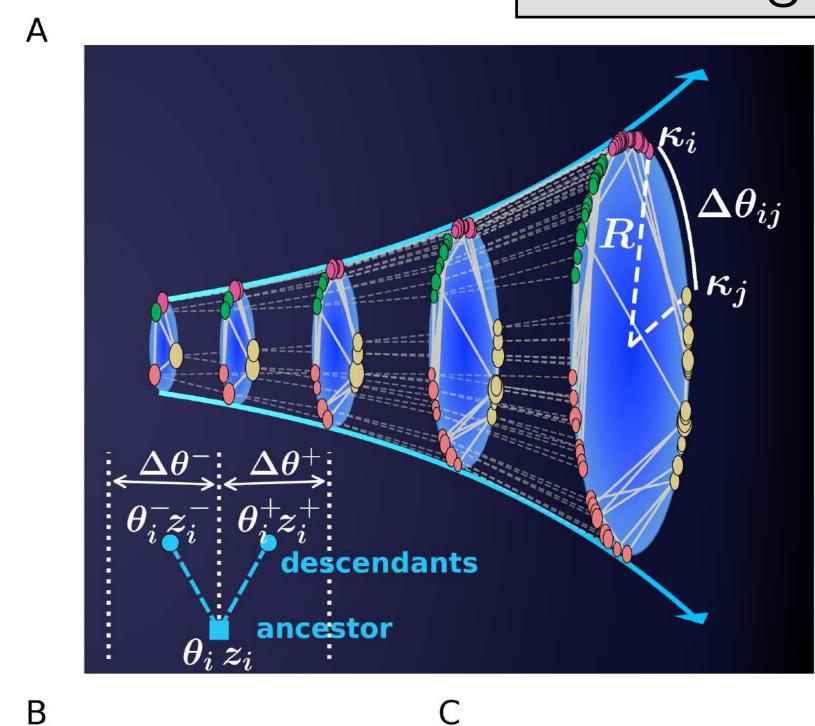
Uncover the self-similar architecture of complex networks
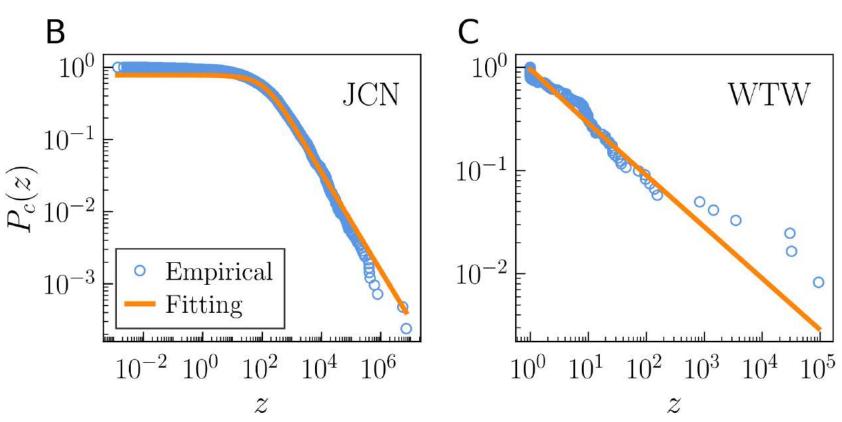
Geometric renormalization group

Scaling up real networks by geometric branching growth
M. Zheng, G. García-Pérez, M.Boguñá, M. A. Serrano
PNAS (in press) (2021)

Self-similar branching growth
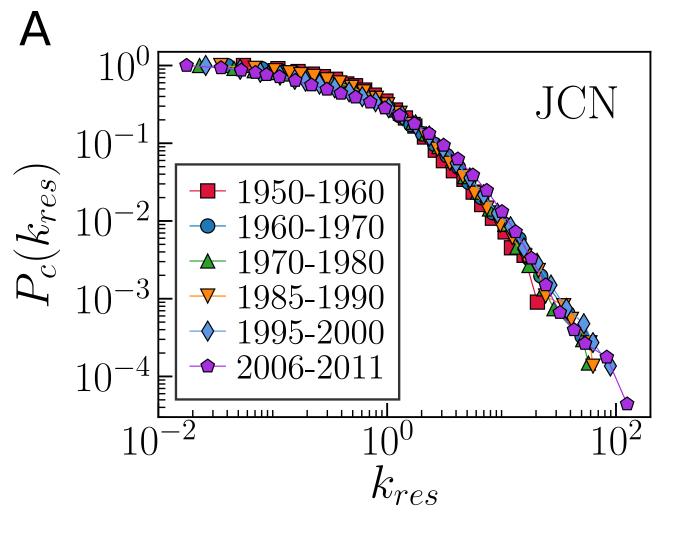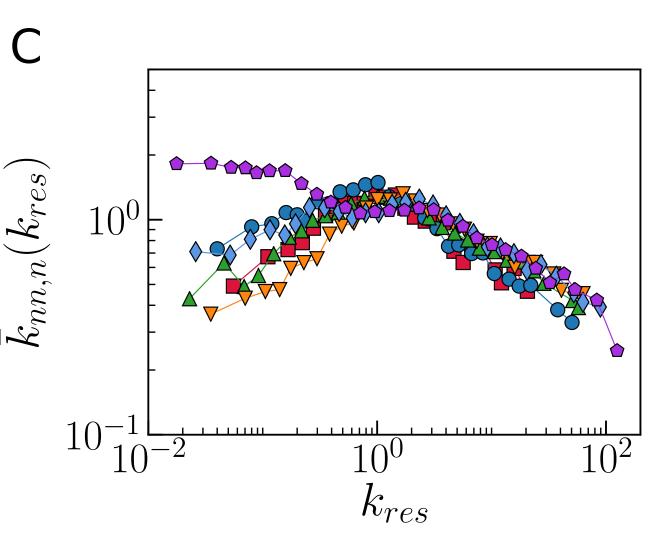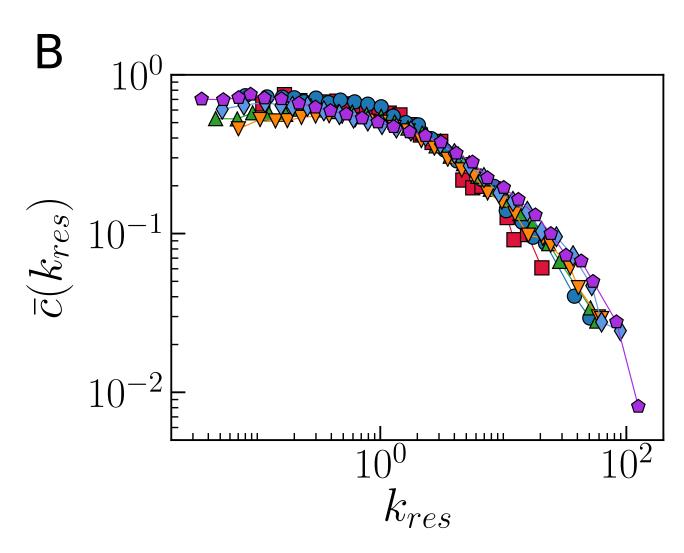
modeling the evolution of complex networks
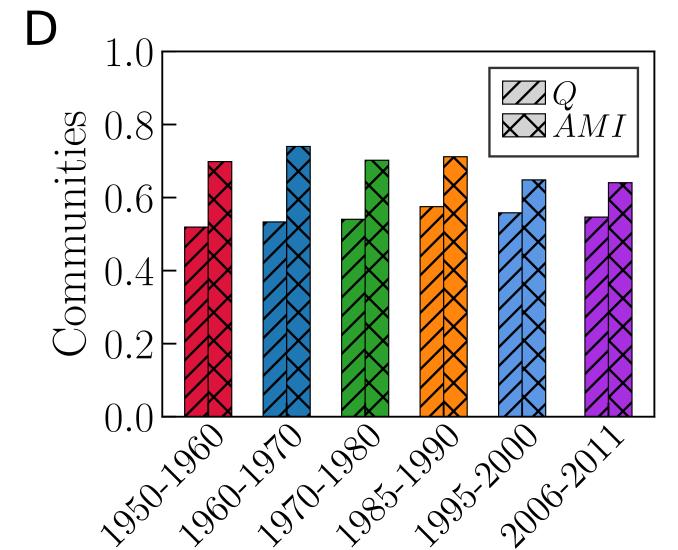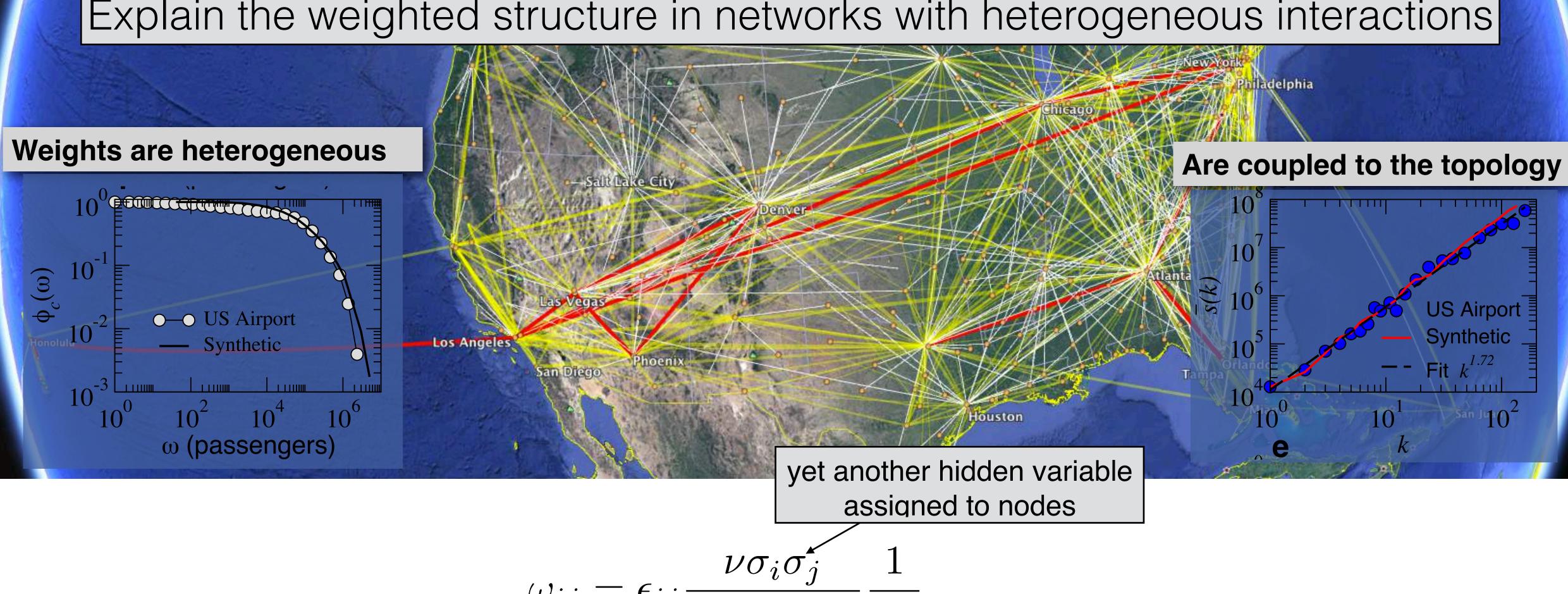
Explain the weighted structure in networks with heterogeneous interactions

Weights are heterogeneous

Are coupled to the topology

yet another hidden variable
assigned to nodes
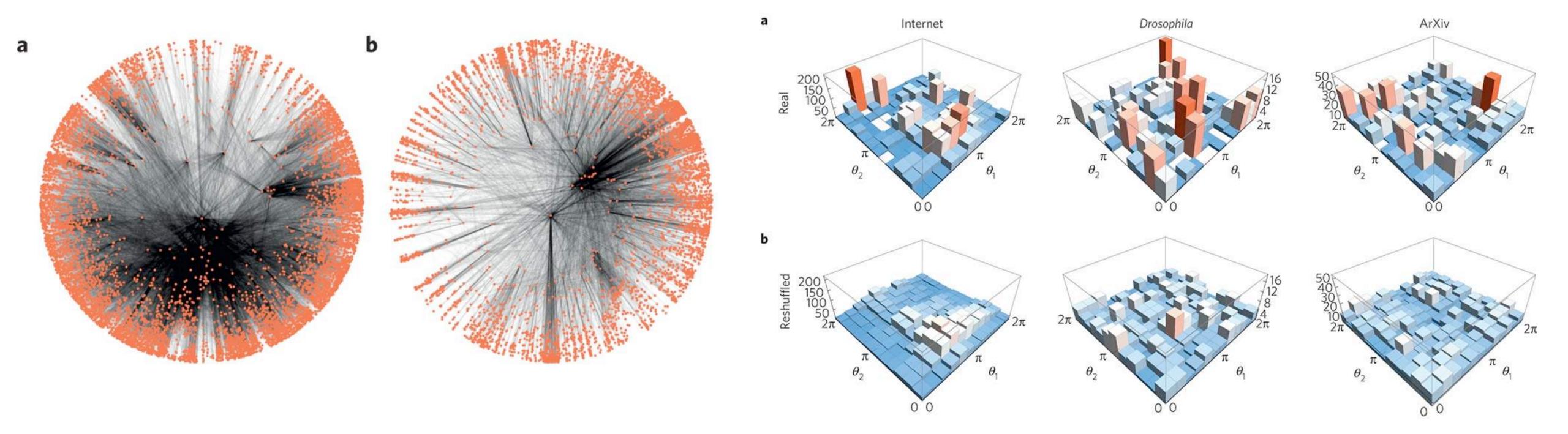
$$\omega_{ij} = \epsilon_{ij} \frac{\nu \sigma_i \sigma_j}{(\kappa_i \kappa_j)^{1-\alpha}} \frac{1}{d_{ij}^{\alpha}}$$

Normalized noise

Coupling parameter

Find correlations between layers in multiplexes

Internet IPv4 layer   Internet IPv6 layer

But are these class of models the only possibility?

Why should the connection probability be of the form?

$$\mathrm{Prob}(m, m', d) = r\left(\frac{d}{mm'}\right)$$

Is hyperbolic geometry the only choice?
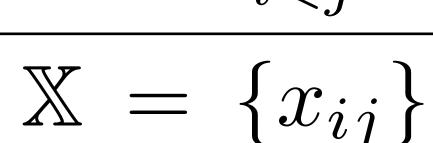
$$S = -\sum_{\mathbb{A}} P(\mathbb{A}) \ln P(\mathbb{A})$$

Entropy of the graph ensemble

$$\langle F_l \rangle = \sum_{\mathbb{A}} F_l(\mathbb{A}) P(\mathbb{A})$$

Constraints
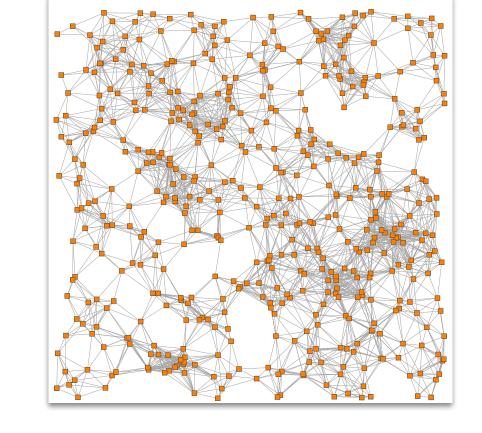
$$P(\mathbb{A}) = \frac{e^{-\sum_l \alpha_l F_l(\mathbb{A})}}{Z}$$

Exponential random graphs

$$Z = \sum_{\mathbb{A}} e^{-\sum_l \alpha_l F_l(\mathbb{A})}$$

Partition function

$$M(\mathbb{A}) = \sum_{i<j} a_{ij}$$

Total number of links

$$\mathbb{X} = \{x_{ij}\}$$

Distances among pairs of nodes

$$\varepsilon_{ij} = f(x_{ij})$$

Energy of a link, as a function of the distance

$$E(\mathbb{A}) = \sum_{i<j} \varepsilon_{ij} a_{ij} = \sum_{i<j} f(x_{ij}) a_{ij}$$

Total energy

Constraints

$$\langle E \rangle = \bar{E},$$
$$\langle M \rangle = \bar{M}$$

$$P(\mathbb{A}) = \frac{\prod_{i<j} e^{\beta(\mu - \varepsilon_{ij})a_{ij}}}{Z} = \prod_{i<j} p_{ij}^{a_{ij}} (1 - p_{ij})^{1-a_{ij}}$$

$$p_{ij} = \frac{1}{e^{\beta(\varepsilon_{ij} - \mu)} + 1}$$

System of non-interacting Fermions in a thermal bath that can occupy any of n(n-1)/2 posible states of energies

$$\varepsilon_{ij} = f(x_{ij})$$

Energy of a link, as a function of the distance

Small worlds and clustering in spatial networks
M. Boguñá, D. Krioukov, P. Almagro, M. Á. Serrano
Physical Review Research 2 (2), 023040 (2020)

Homogeneous geometric random graphs

We look for networks that are simultaneously

Sparse $\quad \langle k \rangle = \lim_{n \to \infty} \sum_j p_{ij} \;$ = Constant

Clustered $\quad p_{ij} = \dfrac{1}{e^{\beta(\varepsilon_{ij} - \mu)} + 1} \quad$ Size independent

Small-world $\quad \lim_{n \to \infty} \left\langle \sum_j x_{ij}^m a_{ij} \right\rangle = \infty$

There is only one possibility meeting all three requirements

$$\epsilon_{ij} = f(x_{ij}) \sim \ln x_{ij} \qquad \beta \in [d, 2d]$$

Dimension of the space

$$p_{ij} \sim \frac{1}{x_{ij}^{\beta}}$$

Parameter regime	Small world	Clustering
$\beta \to 0$ (ER) $\beta < d$	Yes	No
$d \quad < \beta < 2d$	Yes	Yes
$\beta > 2d$ $\beta \to \infty$ (RGG)	No	Yes

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Heterogeneous geometric random graphs

We look for networks that are simultaneously

$$\boxed{\text{Sparse}} \qquad \left\langle \sum_j a_{ij} \right\rangle = \kappa_i$$

$$\boxed{\text{Clustered}} \quad p_{ij} = \frac{1}{e^{\beta f(x_{ij}) + \alpha_i + \alpha_j} + 1} \quad \text{Size independent}$$

$$\boxed{\text{Small-world}} \qquad \lim_{n \to \infty} \left\langle \sum_j x_{ij}^m a_{ij} \right\rangle = \infty$$

$$\boxed{\begin{array}{c} f(x_{ij}) \sim \ln x_{ij} \\ \hline \beta \in [d, 2d] \end{array}} \; + \; \boxed{\text{Degree-degree uncorrelated}} \\ \boxed{f(x_{ij}) = \ln x_{ij} \;\; \forall x_{ij} \geq 0}$$

The connection probability takes a Fermi form

$$p_{ij} = \frac{1}{e^{\beta(\varepsilon_{ij} - \mu)} + 1}$$

With redefined energies of states

$$\epsilon_{ij} = \ln\left[\frac{x_{ij}}{(\kappa_i \kappa_j)^{\frac{1}{d}}}\right] \text{ and } \mu = \frac{1}{d}\ln\hat{\mu}$$

The connection probability becomes the connection probability of the S¹/H² model

$$p_{ij} = \frac{1}{1 + \left[\dfrac{x_{ij}}{(\hat{\mu}\kappa_i\kappa_j)^{\frac{1}{d}}}\right]^{\beta}}$$

The take home lesson

The S^1/H^2 model with the Fermi connection probability is the only maximally random ensemble of geometric random graphs that are simultaneously

Sparse with arbitrary degree distribution

Clustered in the thermodynamic limit

Small-world   Without degree-degree correlations

Thank you very much for your attention

Self-Similarity of Complex Networks and Hidden Metric Spaces
M. A. Serrano, D. Krioukov, and M. Boguñá
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Guillermo García-Pérez, Marián Boguñá, and M. Ángeles Serrano
Nature Physics doi:10.1038/s41567-018-0072-5 (2018)

Mercator: uncovering faithful hyperbolic embeddings of complex networks
G. García-Pérez, A. Allard, M. Á. Serrano, M. Boguñá
New Journal of Physics 21 (12), 123033 (2019)

Small worlds and clustering in spatial networks
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Network geometry
M. Boguñá, I. Bonamassa, M. De Domenico, S. Havlin, D. Krioukov, M. Ángeles Serrano
Nature Reviews Physics 3, 114–135(2021)

Scaling up real networks by geometric branching growth
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